# The Shape of Compact Toroidal Dimensions $T_\theta^d$ and the Casimir Effect on $M^D \times T_\theta^d$ spacetime

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**Abstract**

We study the influence of the shape of compact dimensions to the Casimir energy and Casimir force of a scalar field. We examine both the massive and the massless scalar field. The total spacetime topology is $M^D \times T_\theta^2$, where $M^D$ is the $D$ dimensional Minkowski spacetime and $T_\theta^2$ the twisted torus described by $R_1$, $R_2$ and $\theta$. For the case $R_1 = R_2$ we found that the massive bulk scalar field Casimir energy is singular for $D$=even and this singularity is $R$-dependent and remains even when the force is calculated. Also the massless Casimir energy and force is regular only for $D = 4$ (!). This is very interesting phenomenologically. We examine the energy and force as a function of $\theta$. Also we address the stabilization problem of the compact space. We also briefly discuss some phenomenological implications.

# Introduction

The Casimir effect is one of the many macroscopic manifestations of quantum fluctuations. Since the original paper of H. Casimir the computation of the Casimir energy and Casimir force has developed to a research area on its own, with many theoretical and experimental applications [10]. The applications are vast, varying from the calculation of the vacuum energy between plates to cosmological implications. In most cases the Casimir energy is affected from the geometry and topology of the spacetime.

Many studies have focused on the calculation of the Casimir energy in the presence of compactified space dimensions, see for example [11]. The main interest is focused on the sign of the Casimir energy and Casimir force. Concerning the stabilization of the compact dimensions, there exist many approaches in these issues. Some of these deal with the stabilization of the compact extra dimensions through the radion field. When

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the radion field acquires positive mass square due to a repulsive (positive) Casimir force. The last requires the presence of negative tension brane [7]. A negative mass square is due to attractive Casimir force and corresponds to unstable minimum of the radion [7]. Also calculations have been performed in the presence of compact non-commutative extra dimensions. The interest on these calculations is focused mainly on the calculation of the Casimir force with respect to the compact space. A negative Casimir energy is a very good feature of these theories since it leads to a shrinking of the compact dimensions. Also in some cases one loop corrections lead to stabilization of the compact space. We shall discuss on this more in the following. In addition, calculation of the Casimir energy poses restrictions to the size of the extra dimensions, see for example [9, 6]. However in most studies where compact dimensions are taken into account, the main interest concerning the compact space is focused on the volume of the extra dimensions. With volume we mean the size of the radii of the compact dimensions. Less interest has been given on the shape of the extra dimensions. In the papers [2] of K. Dienes, the study was focused on the effect of the shape of the compact space in the phenomenology of four dimensional spacetime. Also in the paper of K. Kirsten and E. Elizalde [4] the calculation of the Casimir energy for an arbitrary shaped two dimensional toroidal surface was firstly performed. The results found by K. Dienes are very interesting. Specifically it seems that the shape of extra dimensions induce level crossings and varying mass gaps, the elimination of light KK states and the fascinating possibility of the alteration of the experimental constraints for the extra dimensions. Also the "shadowing" process is a very interesting feature. K. Dienes deals with a torus with twisted lattice, which we refer here as twisted torus and we denote it $T^2_\theta$ for brevity. The twisted torus can be seen in Fig.1. The parameters describing the twisted torus are $R_1$, $R_2$ and $\theta$.

In this article we shall include the effect of the shape of compact dimensions to the Casimir energy. Our aim is to study the Casimir energy and Casimir force for spacetime topologies $M^D \times T^2_\theta$, with $M^D$ Minkowski space times and $T^2_\theta$ the arbitrary shaped twisted two dimensional torus. The study will be for the scalar field quantized with this topology, both with and without mass. We are interested to see how the Casimir energy and the corresponding Casimir force behaves as a function of the shape of the extra dimensions parameter (which is $\theta$, see below). This study will include the Casimir force sign and we check if the form of the Casimir energy leads to a stabilization of the compact space. Additionally we shall focus where applicable to our four dimensional spacetime, which is the most interesting case phenomenologically.

In section 1 we describe briefly the eigenfunctions and eigenvalues of the scalar field for the $M^D \times T^2_\theta$ topology, following [2]. In section 2 we compute the Casimir energy and force for a massive scalar on $M^D \times T^2_\theta$. In section 3 we do the same for the massless scalar. The conclusions with a discussion follow in section 4.

# 1 Eigenfunctions and Eigenvalues on $M^D \times T^2_\theta$

We describe here the twisted torus $T^2_\theta$ and the eigenfunctions and eigenvalues of the scalar field for the spacetime $M^D \times T^2_\theta$. We shall follow the presentation of K. Dienes [2].

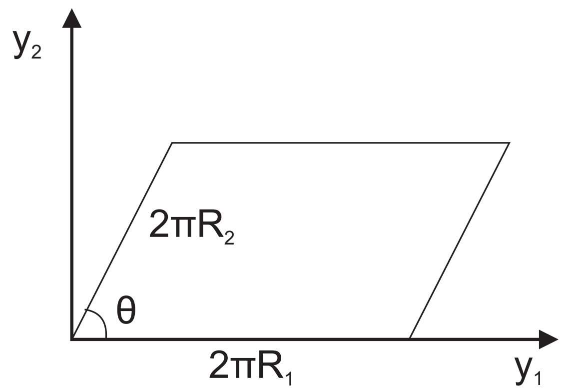


Figure 1: The twisted torus described by $R_1$, $R_2$ and $\theta$.

The general twisted torus case described by Figure 1 with the torus radii $R_1$ and $R_2$ and also with lattice angle $\theta$. The twisted torus $T_\theta^2$ is realized as the flat space $R^2$ described by the coordinates $y_1$ and $y_2$ with the identifications,

$$\begin{aligned} y_1 &\to y_1 + 2\pi R_1 \\ y_2 &\to y_2 \\ y_1 &\to y_1 + 2\pi R_2 \cos\theta \\ y_2 &\to y_2 + 2\pi R_2 \sin\theta \end{aligned} \tag{1}$$

It is obvious that the value $\theta = \frac{\pi}{2}$ gives the known torus $T^2$. Our aim here is to present the solutions to the Laplace equation,

$$-\Delta_{M^D \times T_\theta^2} \phi(x, y_1, y_2) = \omega_{M^D \times T_\theta^2} \phi(x, y_1, y_2) \tag{2}$$

for the scalar field in the spacetime $M^D \times T_\theta^2$ and also the eigenvalues (where $x$ denotes the space coordinates of the $M^D$ spacetime coordinate). Demanding invariance under the torus identifications (1), the eigenfunctions are,

$$\phi(x, y_1, y_2) = \int \frac{\mathrm{d}^{D-1}p}{(2\pi)^{D-1}} \sum_{n,n_1=-\infty}^{\infty} e^{i\vec{p}\vec{x}} e^{i\frac{n}{R_1}\left(y_1 - \frac{y_2}{\tan\theta}\right) + i\frac{n_1\, y_2}{R_2 \sin\theta}}. \tag{3}$$

Now it follows that the eigenvalues for the massive scalar are,

$$\omega_{M^D \times T_\theta^2} = \sum_{k=1}^{D-1} p_k^2 + \frac{n^2}{\sin^2\theta R_1^2} + \frac{n_1^2}{\sin^2\theta R_2^2} - 2\frac{nn_1}{R_1 R_2}\cos\theta + m^2, \tag{4}$$

and for the massless,

$$\omega_{M^D \times T_\theta^2} = \sum_{k=1}^{D-1} p_k^2 + \frac{n^2}{\sin^2\theta R_1^2} + \frac{n_1^2}{\sin^2\theta R_2^2} - 2\frac{nn_1}{R_1 R_2}\cos\theta. \tag{5}$$

The $\theta$ values are restricted to the range $0 < \theta \leq \frac{\pi}{2}$ without loss of generality. We shall use the following notation in the next sections, namely,

$$a = \frac{1}{\sin^2\theta R_1^2} \qquad (6)$$
$$c = \frac{1}{\sin^2\theta R_2^2}$$
$$b = -2\frac{1}{R_1R_2}\cos\theta$$

and also,

$$\Delta = 4ac - b^2 = \frac{4}{R_1^2R_2^2\sin^2\theta}\left(1 - \sin^2\theta\cos^2\theta\right) \qquad (7)$$

For later use note that $\Delta \geq 0$ for all $\theta$ values. Our interest is mainly for the case $R_1 = R_2$. Thus we shall try to find how the changes of $\theta$ alter the Casimir energy. However in the case with equal torus radii the shadowing phenomenon is absent. We shall discuss on these issues in the conclusions.

# 2 Casimir energy and Casimir force for the massive scalar field on $M^D \times T_\theta^2$

## 2.1 Casimir energy with general $R_1$, $R_2$ and $\theta$

We now calculate the Casimir energy for the massive scalar in the $M^D \times T_\theta^2$ spacetime. Using relation (4) and the notation of relations (6) and (7), the Casimir energy for the twisted torus reads (at the end we put $s = -\frac{1}{2}$),

$$\mathcal{E}_c(s) = \frac{1}{(2\pi)^{D-1}}\int \mathrm{d}^{D-1}p \sum_{n,n_1=-\infty}^{\infty}\Big[\sum_{k=1}^{D-1}p_k^2 + an^2 + b\,nn_1 + cn_1^2 + m^2\Big]^{-s}. \qquad (8)$$

In the end we put $s = -\frac{1}{2}$. Upon integrating over the continuous dimensions using,

$$\int \mathrm{d}k^{D-1}\frac{1}{(k^2+A)^s} = \pi^{\frac{D-1}{2}}\frac{\Gamma(s-\frac{D-1}{2})}{\Gamma(s)}\frac{1}{A^{s-\frac{D-1}{2}}} \qquad (9)$$

relation (8) becomes,

$$\mathcal{E}_c(s,a) = \frac{1}{(2\pi)^{D-1}}\pi^{\frac{D-1}{2}}\frac{\Gamma(s-\frac{D-1}{2})}{\Gamma(s)}\sum_{n,n_1=-\infty}^{\infty}\Big[an^2 + b\,nn_1 + cn_1^2 + m^2\Big]^{\frac{D-1}{2}-s}. \qquad (10)$$

Using the inhomogeneous Epstein zeta-like function [3, 11, 4, 13, 14],

$$E(s;a,b,c;q) = \sum_{n,n_1=-\infty}^{\infty\,\prime}\Big[an^2 + b\,nn_1 + cn_1^2 + m^2\Big]^{-s} \qquad (11)$$

relation (12) is written,

$$
\mathcal{E}_c(s,a) = \frac{1}{(2\pi)^{D-1}}\pi^{\frac{D-1}{2}}\frac{\Gamma(s-\frac{D-1}{2})}{\Gamma(s)}\ E(s-\frac{D-1}{2};a,b,c;m^2) \tag{12}
$$

$$
+\frac{1}{(2\pi)^{D-1}}\pi^{\frac{D-1}{2}}\frac{\Gamma(s-\frac{D-1}{2})}{\Gamma(s)}\,(m^2)^{-(s-\frac{D-1}{2})}.
$$

Now the inhomogeneous can be expanded according to the extended Chowla-Selberg formula [3, 4],

$$
E(s;a,b,c;q) = 2\,\zeta_{EH}(s,\frac{q}{a})\,a^{-s} + \frac{2^{2s}\sqrt{\pi}\,a^{s-1}}{\Gamma(s)\Delta^{s-\frac{1}{2}}}\Gamma(s-\frac{1}{2})\zeta_{EH}(s-\frac{1}{2},\frac{4aq}{\Delta}) \tag{13}
$$

$$
+\frac{2^{s+\frac{5}{2}}\,\pi^s}{\Gamma(s)\sqrt{a}}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\cos(\frac{n\pi b}{a})\sum_{d/n}d^{1-2s}\Big(\Delta+\frac{4aq}{d^2}\Big)^{-\frac{s}{2}+\frac{1}{4}}K_{s-\frac{1}{2}}\Big(\frac{\pi n}{a}\sqrt{\Delta+\frac{4aq}{d^2}}\Big)
$$

which is defined for $\Delta \geq 0$ which in our case holds as we saw previously. In the above $\zeta_{EH}(s;p)$ stands for the inhomogeneous Epstein zeta [11, 3, 4, 12, 13, 14],

$$
\zeta_{EH}(s;p) = \frac{1}{2}\sum_{n=-\infty}^{\infty\,\prime}\big(n^2+p\big)^{-s} \tag{14}
$$

$$
= -\frac{p^{-s}}{2} + \frac{\sqrt{\pi}\ \Gamma(s-\frac{1}{2})}{2\Gamma(s)}\,p^{-s+\frac{1}{2}} + \frac{2\pi^s\,p^{-s+\frac{1}{2}}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\big(2\pi n\sqrt{p}\big)
$$

Thus with the help of (13), the Casimir energy of (12) can be written,

$$
\mathcal{E}_c(s) = \frac{1}{(2\pi)^{D-1}}\pi^{\frac{D-1}{2}}\frac{\Gamma(s-\frac{D-1}{2})}{\Gamma(s)}\ \times \tag{15}
$$

$$
\Bigg(2\,\zeta_{EH}(s-\frac{D-1}{2},\frac{q}{a})\,a^{-(s-\frac{D-1}{2})} + \frac{2^{2(s-\frac{D-1}{2})}\sqrt{\pi}\,a^{s-\frac{D-1}{2}-1}}{\Gamma(s-\frac{D-1}{2})\,\Delta^{s-\frac{D-1}{2}-\frac{1}{2}}}\times
$$

$$
\Gamma(s-\frac{D-1}{2}-\frac{1}{2})\ \zeta_{EH}(s-\frac{D-1}{2}-\frac{1}{2},\frac{4aq}{\Delta})
$$

$$
+\frac{2^{s-\frac{D-1}{2}+\frac{5}{2}}\,\pi^{s-\frac{D-1}{2}}}{\Gamma(s-\frac{D-1}{2})\sqrt{a}}\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-\frac{1}{2}}\cos(\frac{n\pi b}{a})\times
$$

$$
\sum_{d/n}d^{1-2(s-\frac{D-1}{2})}\Big(\Delta+\frac{4aq}{d^2}\Big)^{-\frac{s-\frac{D-1}{2}}{2}+\frac{1}{4}}K_{s-\frac{D-1}{2}-\frac{1}{2}}\Big(\frac{\pi n}{a}\sqrt{\Delta+\frac{4aq}{d^2}}\Big)\Bigg)
$$

$$
+\frac{1}{(2\pi)^{D-1}}\pi^{\frac{D-1}{2}}\frac{\Gamma(s-\frac{D-1}{2})}{\Gamma(s)}\,(m^2)^{-(s-\frac{D-1}{2})}.
$$

Now let us write each term of relation (15) separately, in order to present the details of

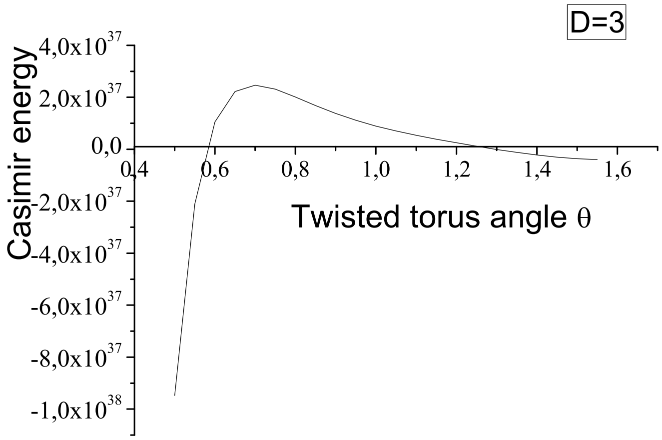


Figure 2: The Casimir energy $\mathcal{E}_c$ as a function of $\theta$, the twisted torus angle, for $D = 3$. (Massive case)

the calculations. The first term is written,

$$\begin{aligned}
&2\,\zeta_{EH}(s-\frac{D-1}{2},\frac{q}{a})\,a^{-(s-\frac{D-1}{2})} = -(\frac{q}{a})^{-(s-\frac{D-1}{2})}a^{-(s-\frac{D-1}{2})} \\
&+ a^{-(s-\frac{D-1}{2})}\frac{\sqrt{\pi}\,\Gamma(s-\frac{D-1}{2}-\frac{1}{2})}{\Gamma(s-\frac{D-1}{2})}\,(\frac{q}{a})^{-(s-\frac{D-1}{2})+\frac{1}{2}} \\
&+ a^{-(s-\frac{D-1}{2})}(\frac{q}{a})^{-\frac{1}{2}(s-\frac{D-1}{2})+\frac{1}{4}}\frac{2\,\pi^{s-\frac{D-1}{2}}}{\Gamma(s-\frac{D-1}{2})}\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-\frac{1}{2}}K_{s-\frac{D-1}{2}-\frac{1}{2}}\big(2\pi n\sqrt{\frac{q}{a}}\big)
\end{aligned} \tag{16}$$

The second term is written,

$$\begin{aligned}
&\frac{2^{2(s-\frac{D-1}{2})}\sqrt{\pi}\,a^{s-\frac{D-1}{2}-1}}{\Gamma(s-\frac{D-1}{2})\,\Delta^{s-\frac{D-1}{2}-\frac{1}{2}}}\Gamma(s-\frac{D-1}{2}-\frac{1}{2})\;\zeta_{EH}(s-\frac{D-1}{2}-\frac{1}{2},\frac{4aq}{\Delta}) = \\
&-\frac{2^{2(s-\frac{D-1}{2})}\sqrt{\pi}\,a^{s-\frac{D-1}{2}-1}\Gamma(s-\frac{D-1}{2}-\frac{1}{2})}{2\,\Gamma(s-\frac{D-1}{2})\,\Delta^{s-\frac{D-1}{2}-\frac{1}{2}}}(\frac{4aq}{\Delta})^{-(s-\frac{D-1}{2}-\frac{1}{2})} \\
&+\frac{2^{2(s-\frac{D-1}{2})}\pi\,a^{s-\frac{D-1}{2}-1}\Gamma(s-\frac{D-1}{2}-1)}{4\,\Gamma(s-\frac{D-1}{2})\,\Delta^{s-\frac{D-1}{2}-\frac{1}{2}}}(\frac{4aq}{\Delta})^{-(s-\frac{D-1}{2}-\frac{1}{2})+\frac{1}{2}} \\
&+\frac{2^{2(s-\frac{D-1}{2})}\sqrt{\pi}\,a^{s-\frac{D-1}{2}-1}\,\pi^{s-\frac{D-1}{2}-\frac{1}{2}}}{\Gamma(s-\frac{D-1}{2})\,\Delta^{s-\frac{D-1}{2}-\frac{1}{2}}}(\frac{4aq}{\Delta})^{-(s-\frac{D-1}{2}-\frac{1}{2})+\frac{1}{4}}\times \\
&\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-1}K_{s-\frac{D-1}{2}-\frac{1}{2}}\Big(2\pi n\sqrt{\frac{4aq}{\Delta}}\Big)
\end{aligned} \tag{17}$$

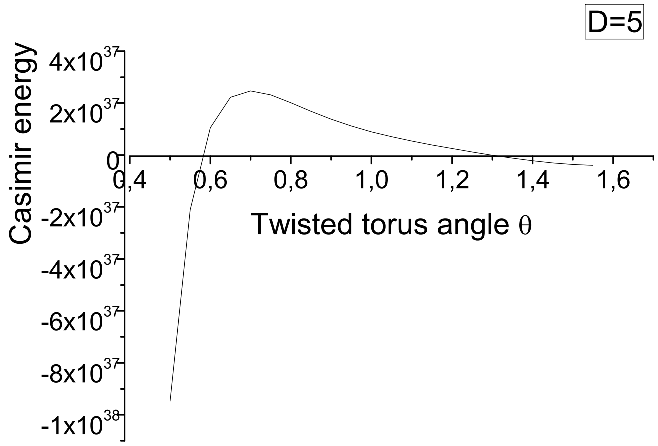


Figure 3: The Casimir energy $\mathcal{E}_c$ as a function of $\theta$, the twisted torus angle, for $D = 5$. (Massive case)

From the above two relations, (16) and (17) it is easily seen that various cancellations occur, for example the first term of the second term cancels the second term of the first term. After some algebra we finally obtain,

$$\mathcal{E}_c(s) = \frac{1}{(2\pi)^{D-1}} \frac{\pi^{\frac{D-1}{2}}}{\Gamma(s)} \times \tag{18}$$

$$\Bigg( 2\,\pi^{s-\frac{D-1}{2}}\, a^{\frac{1}{2}(s-\frac{D-1}{2})+\frac{1}{4}}\, q^{-\frac{1}{2}(s-\frac{D-1}{2})+\frac{1}{4}} \sum_{n=1}^{\infty} n^{s-\frac{D-1}{2}-\frac{1}{2}} K_{s-\frac{D-1}{2}-\frac{1}{2}}(2\pi n \sqrt{\frac{q}{a}})$$

$$+ \frac{2^{2(s-\frac{D-1}{2})} \pi\, a^{s-\frac{D-1}{2}-1} \Gamma(s - \frac{D-1}{2} - 1)}{4\,\Delta^{s-\frac{D-1}{2}-\frac{1}{2}}} (\frac{4aq}{\Delta})^{-(s-\frac{D-1}{2}-\frac{1}{2})+\frac{1}{2}}$$

$$+ \frac{2^{2(s-\frac{D-1}{2})} \sqrt{\pi}\, a^{s-\frac{D-1}{2}-1}\, \pi^{s-\frac{D-1}{2}-\frac{1}{2}}}{\Delta^{s-\frac{D-1}{2}-\frac{1}{2}}} (\frac{4aq}{\Delta})^{-(s-\frac{D-1}{2}-\frac{1}{2})+\frac{1}{4}} \times$$

$$\sum_{n=1}^{\infty} n^{s-\frac{D-1}{2}-1} K_{s-\frac{D-1}{2}-\frac{1}{2}}(2\pi n \sqrt{\frac{4aq}{\Delta}})$$

$$+ \frac{2^{s-\frac{D-1}{2}+\frac{5}{2}}\, \pi^{s-\frac{D-1}{2}}}{\sqrt{a}} \sum_{n=1}^{\infty} n^{s-\frac{D-1}{2}-\frac{1}{2}} \cos(\frac{n\pi b}{a}) \times$$

$$\sum_{d/n} d^{1-2(s-\frac{D-1}{2})} \Big( \Delta + \frac{4aq}{d^2} \Big)^{-\frac{s-\frac{D-1}{2}}{2}+\frac{1}{4}} K_{s-\frac{D-1}{2}-\frac{1}{2}} \Big( \frac{\pi n}{a} \sqrt{\Delta + \frac{4aq}{d^2}} \Big) \Bigg)$$

We can easily see that relation (27) contains a singularity when $D$=even. Indeed the

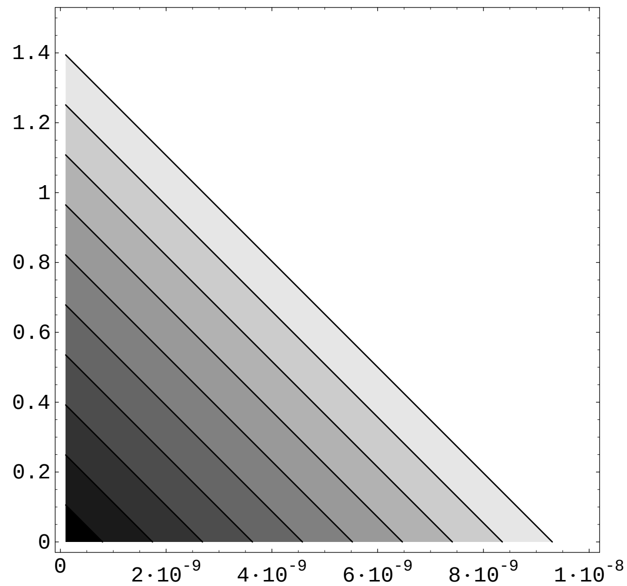


Figure 4: Contour plot of the Casimir energy $\mathcal{E}_c$ as a function of $\theta$ (vertical), the twisted torus angle, and $R$, the compact radius (horizontal), for $D = 3$. (Massive case)

singularity is due to the gamma function $\Gamma(s - \frac{D-1}{2} - 1)$. Thus our four dimensional spacetime is excluded from the study. However we will continue to present the results because these are very interesting mathematically and also in order to have a clear picture for all cases.

## 2.2 The case $R_1 = R_2$

Now we specify our result (27) to the case $R_1 = R_2 = R$. Thus the only parameter that characterizes the shape is the $\theta$ angle of the twisted torus lattice. Within this approxima-

tion the Casimir energy reads,

$$
\begin{aligned}
&\mathcal{E}_c(s) = \frac{1}{(2\pi)^{D-1}}\frac{\pi^{\frac{D-1}{2}}}{\Gamma(s)}\times \qquad (19)\\
&\Bigg(2\,\pi^{s-\frac{D-1}{2}}\,(R\sin\theta)^{-(s-\frac{D-1}{2})-\frac{1}{2}}\,(m^2)^{-\frac{1}{2}(s-\frac{D-1}{2})+\frac{1}{4}}\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-\frac{1}{2}}K_{s-\frac{D-1}{2}-\frac{1}{2}}\big(2\pi n\,mR\sin\theta\big)\\
&+4\pi\,(m^2)^{-(s-\frac{D-1}{2})+1}(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{2}}\Gamma(s-\frac{D-1}{2}-1)\sin\theta R^2\\
&+2^{3/2}\,(m^2)^{-(s-\frac{D-1}{2})+\frac{1}{4}}(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{4}}\,\pi^{s-\frac{D-1}{2}}R^{\frac{3}{2}}\sin\theta\times\\
&\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-1}K_{s-\frac{D-1}{2}-\frac{1}{2}}\big(4\pi n\,mR(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{2}}\big)\\
&+2^{s-\frac{D-1}{2}+\frac{5}{2}}\,\pi^{s-\frac{D-1}{2}}R\sin\theta\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-\frac{1}{2}}\cos(2n\pi\cos\theta\sin^2\theta)\times\\
&\sum_{d/n}d^{1-2(s-\frac{D-1}{2})}\Big(\frac{(1-\cos^2\theta\sin^2\theta)}{R^4\sin^2\theta}+\frac{4m^2}{R^2\sin^2\theta d^2}\Big)^{-\frac{1}{2}(s-\frac{D-1}{2})+\frac{1}{4}}\times\\
&K_{s-\frac{D-1}{2}-\frac{1}{2}}\Big(n\pi R\sin\theta\sqrt{\frac{(1-\cos^2\theta\sin^2\theta)}{R^2}+\frac{4m^2}{d^2}}\,\Big)\Bigg)
\end{aligned}
$$

Of course $D$=odd. We shall study the cases $D=3$ and $D=5$ but the results are similar, really interesting and very permissible for theories with compact extra dimensions.
In Figure 2 we plot the Casimir energy for $D=3$, $R=10^{-8}$ and $m=100$ and in Figure 3 for $D=5$. Also in Figure 4 we present the contour plot of the Casimir energy as a function of $R$ and $\theta$. The lighter colors correspond to larger values of the Casimir energy. Let us discuss on these. As we can see the Casimir energy for both $D$=3 and $D$=5 takes negative and positive values, with varying $\theta$. The negative values of the Casimir energy is a very attractive feature of theories with compact extra dimensions. Notice that near the most studied case $\theta=\frac{\pi}{2}$ the Casimir energy is negative. When the Casimir energy is negative (and more and more negative as $R$ gets smaller) this leads to a shrinking of the compact dimensions. This is true in the case the Casimir energy contains inverse powers of $R$ and this is our case also. Thus we see that near $\theta=\frac{\pi}{2}$, the Casimir energy is negative. Behind this fact is the existence of an attractive Casimir force. We now compute the Casimir force and we continue soon this discussion. The computation of the Casimir force from (19) is straightforward. The Casimir force equals to,

$$\mathcal{F}_c=-\frac{\partial\mathcal{E}_c}{\partial R} \qquad (20)$$

and using

$$\frac{\partial}{\partial x}K_\nu(x\,z)=-\frac{1}{4}z\,K_{\nu-1}(x\,z)-\frac{1}{4}z\,K_{\nu+1}(x\,z) \qquad (21)$$

the Casimir force for the massive case reads,

$$
\begin{aligned}
&\mathcal{F}_c(s) = -\frac{1}{(2\pi)^{D-1}}\frac{\pi^{\frac{D-1}{2}}}{\Gamma(s)}\times \qquad (22)\\
&\Bigg(2\,\pi^{s-\frac{D-1}{2}}\,(\sin\theta)^{\frac{1}{2}-(s-\frac{D-1}{2})}(-(s-\frac{D-1}{2})-\frac{1}{2})\times\\
&R^{-(s-\frac{D-1}{2})-\frac{3}{2}}\,(m^2)^{-\frac{1}{2}(s-\frac{D-1}{2})+\frac{1}{4}}\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-\frac{1}{2}}K_{s-\frac{D-1}{2}-\frac{1}{2}}\big(2\pi n\,mR\sin\theta\big)\\
&-\frac{1}{2}\pi^{s-\frac{D-1}{2}}\,(R\sin\theta)^{-(s-\frac{D-1}{2})-\frac{1}{2}}\,(m^2)^{-\frac{1}{2}(s-\frac{D-1}{2})+\frac{1}{4}}2\pi\,m\sin\theta\times\\
&\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}+\frac{1}{2}}\Big\{K_{s-\frac{D-1}{2}+\frac{1}{2}}\big(2\pi n\,mR\sin\theta\big)+K_{s-\frac{D-1}{2}-\frac{3}{2}}\big(2\pi n\,mR\sin\theta\big)\Big\}\\
&+8\pi\,(m^2)^{-(s-\frac{D-1}{2})+1}\Gamma(s-\frac{D-1}{2}-1)\sin\theta R(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{2}}\\
&+2^{3/2}\,(m^2)^{-(s-\frac{D-1}{2})+\frac{1}{4}}(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{4}}\,\pi^{s-\frac{D-1}{2}}\frac{3}{2}R^{\frac{1}{2}}\sin\theta\times\\
&\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-1}K_{s-\frac{D-1}{2}-\frac{1}{2}}\big(4\pi n\,mR(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{2}}\big)\\
&-2^{3/2}\,(m^2)^{-(s-\frac{D-1}{2})+\frac{1}{4}}(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{4}}\,\pi^{s-\frac{D-1}{2}}R^{\frac{3}{2}}\,\pi\,m\sin\theta\times\\
&\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}}\Big\{K_{s-\frac{D-1}{2}-\frac{3}{2}}\big(4\pi n\,mR(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{2}}\big)\\
&+K_{s-\frac{D-1}{2}+\frac{1}{2}}\big(4\pi n\,mR(1-\cos^2\theta\sin^2\theta)^{-\frac{1}{2}}\big)\Big\}\\
&+2^{s-\frac{D-1}{2}+\frac{5}{2}}\,\pi^{s-\frac{D-1}{2}}\sin\theta\sum_{n=1}^{\infty}n^{s-\frac{D-1}{2}-\frac{1}{2}}\cos(2n\pi\cos\theta\sin^2\theta)\times\\
&\sum_{d/n}d^{1-2(s-\frac{D-1}{2})}\Big(\frac{(1-\cos^2\theta\sin^2\theta)}{R^4\sin^2\theta}+\frac{4m^2}{R^2\sin^2\theta d^2}\Big)^{-\frac{1}{2}(s-\frac{D-1}{2})+\frac{1}{4}}\times\\
&\Bigg\{K_{s-\frac{D-1}{2}-\frac{1}{2}}\Big(n\pi R\sin\theta\sqrt{\frac{(1-\cos^2\theta\sin^2\theta)}{R^2}+\frac{4m^2}{d^2}}\Big)\\
&-4\big[\frac{1}{4}-(s-\frac{D-1}{2})\big]K_{s-\frac{D-1}{2}-\frac{1}{2}}\Big(n\pi R\sin\theta\sqrt{\frac{(1-\cos^2\theta\sin^2\theta)}{R^2}+\frac{4m^2}{d^2}}\Big)\\
&-\frac{R}{2}\Big(\sqrt{\frac{(1-\cos^2\theta\sin^2\theta)}{R^2}+\frac{4m^2}{d^2}}\sin\theta\,\pi-\frac{\sin\theta\,\pi}{\sqrt{\frac{(1-\cos^2\theta\sin^2\theta)}{R^2}+\frac{4m^2}{d^2}}\,R^2}\Big)\times\\
&\Big[K_{s-\frac{D-1}{2}+\frac{1}{2}}\Big(n\pi R\sin\theta\sqrt{\frac{(1-\cos^2\theta\sin^2\theta)}{R^2}+\frac{4m^2}{d^2}}\Big)\\
&+K_{s-\frac{D-1}{2}-\frac{3}{2}}\Big(n\pi R\sin\theta\sqrt{\frac{(1-\cos^2\theta\sin^2\theta)}{R^2}+\frac{4m^2}{d^2}}\Big)\Big]\Bigg\}\Bigg)
\end{aligned}
$$

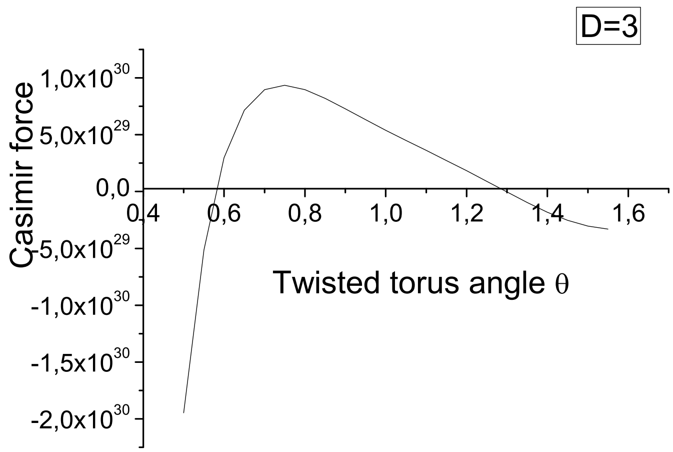


Figure 5: The Casimir force $\mathcal{F}_c$ as a function of $\theta$, the twisted torus angle, for $D = 3$. (Massive case)

The Bessel series is converging really fast. Also in the last term the argument of the Bessel function is $R$-independent. Thus only a few terms of the last term give the dominating contribution. During the process we ascertained that the other Bessel sums are really fast convergent independently of the number of terms we keep. In Figure 5 we plot the Casimir force for $D = 3$, $R = 10^{-8}$ and $m = 100$, and in Figure 6 we present the contour plot of the force as a function of $R$ and $\theta$.
As is seen from Fig. 5, the Casimir force as a function of $\theta$ behaves as the Casimir energy does. As it can be seen the force and energy change sign as $\theta$ varies. Also there exist $\theta$ values for which the Casimir energy is completely zero. The most interesting cases are those for which the energy and force are both negative. Indeed in this case the internal space shrinks without limit. The Casimir force is due to the non trivial topology of the compact space and is responsible for the compact dimensions shrinking. It is necessary for the compact space to be stabilized before it shrinks to very small lengths. However this does not happen here, as it can be easily checked. Indeed there is no stable minimum for the Casimir energy as a function of $R$. We have not checked the case when $R_1 \neq R_2$ but this is out of the scopes of this paper.

## 3 Massless Case

In this section we calculate the scalar Casimir energy and Casimir force for $M^D \times T^2_\theta$. It seems that this case is very interesting phenomenologically since the calculations are valid (that is singularity free) only for $D$=4 (!). This is indeed surprising. Indeed in this case

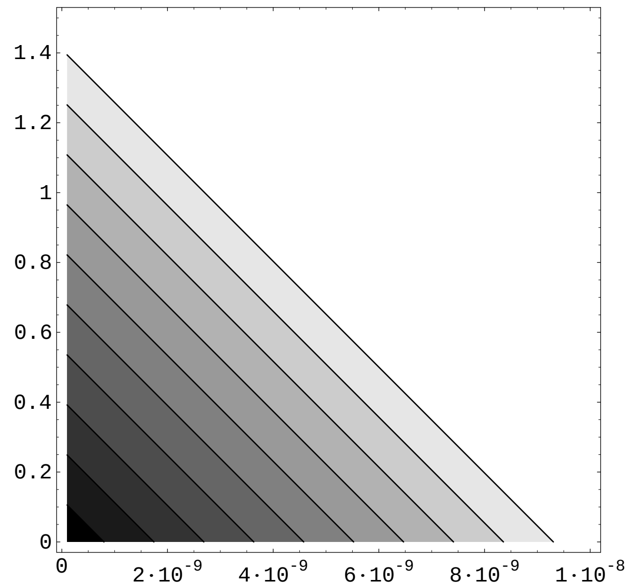


Figure 6: Contour plot of the Casimir energy $\mathcal{E}_c$ as a function of $\theta$ (vertical), the twisted torus angle, and $R$, the compact radius (horizontal), for $D = 3$. (Massive case)

the regularized Casimir energy for the twisted torus reads,

$$\mathcal{E}_c(s) = \frac{1}{(2\pi)^{D-1}} \int \mathrm{d}^{D-1}p \sum_{n,\, n_1=-\infty}^{\infty} \Big[ \sum_{k=1}^{D-1} p_k^2 + an^2 + b\, nn_1 + cn_1^2 \Big]^{-s} \tag{23}$$
$$- \frac{1}{(2\pi)^{D-1}} \int \mathrm{d}^{D-1}p \Big[ \sum_{k=1}^{D-1} p_k^2 \Big]^{-s}.$$

and using (9), we get,

$$\mathcal{E}_c(s) = \frac{1}{(2\pi)^{D-1}} \pi^{\frac{D-1}{2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \sum_{n,\, n_1=-\infty}^{\infty\;\prime} \Big[ an^2 + b\, nn_1 + cn_1^2 \Big]^{\frac{D-1}{2}-s}. \tag{24}$$

Using the homogeneous Epstein zeta-like function,

$$E(s; a, b, c) = \sum_{n,\, n_1=-\infty}^{\infty\;\prime} \Big[ an^2 + b\, nn_1 + cn_1^2 + m^2 \Big]^{-s} \tag{25}$$

and the Chowla-Selberg expansion that holds for it in this case [11, 4],

$$E(s;a,b,c) = 2\,\zeta(2s)a^{-s} + \frac{2^{2s}\sqrt{\pi}\;\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\Delta^{s-\frac{1}{2}}}\,a^{s-1} \tag{26}$$
$$+ \frac{2^{s+\frac{5}{2}}\pi^s}{\Gamma(s)\sqrt{a}\,\Delta^{\frac{s}{2}-\frac{1}{4}}}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sum_{d/n} d^{1-2s}\cos(\frac{n\pi b}{a})K_{s-\frac{1}{2}}(\frac{\pi n\sqrt{\Delta}}{a})$$

the Casimir energy in the massless case is written,

$$\mathcal{E}_c(s) = \frac{1}{(2\pi)^{D-1}}\frac{\pi^{\frac{D-1}{2}}}{\Gamma(s)}\times \tag{27}$$
$$\Bigg(2\,\zeta(2s-D+1)a^{-(s-\frac{D-1}{2})}\Gamma(s-\frac{D-1}{2})$$
$$+\frac{2^{2s-D+1}\sqrt{\pi}\;\Gamma(s-\frac{D-1}{2}-\frac{1}{2})\zeta(2s-D)}{\Delta^{s-\frac{D-1}{2}-\frac{1}{2}}}\,a^{s-\frac{D-1}{2}-1}$$
$$+\frac{2^{s-\frac{D-1}{2}+\frac{5}{2}}\,\pi^{s-\frac{D-1}{2}}}{\Delta^{\frac{1}{2}(s-\frac{D-1}{2})-\frac{1}{4}}\sqrt{a}}\sum_{n=1}^{\infty} n^{s-\frac{D-1}{2}-\frac{1}{2}}\cos(\frac{n\pi b}{a})\sum_{d/n} d^{1-2(s-\frac{D-1}{2})}K_{s-\frac{D-1}{2}-\frac{1}{2}}\Big(\frac{\pi n}{a}\sqrt{\Delta}\Big)\Bigg)$$

As we can see, the Casimir energy is free of divergences only when $D$=4. This is because the combination of the gamma functions of the first two terms gives always a divergent contribution. However for $D$=4 the first term is zero since $\zeta(-4)=0$ (note that $s=-\frac{1}{2}$). It is very surprising that something really works solely for four dimensions and not for other dimensions. We shall concentrate our study for the $R_1 = R_2$ case. It is clear that, as before, the only parameter that characterizes the shape of the extra dimensions is the twisted torus angle, $\theta$.
In the case $R_1 = R_2$ the Casimir energy (27) reads,

$$\mathcal{E}_c(s) = \frac{1}{(2\pi)^{D-1}}\frac{\pi^{\frac{D-1}{2}}}{\Gamma(s)}\times \tag{28}$$
$$\Bigg(2\,\zeta(2s-D+1)(R^2\sin^2\theta)^{(s-\frac{D-1}{2})}\Gamma(s-\frac{D-1}{2})$$
$$+\,2^{2s-D+1}\sqrt{\pi}\,\Gamma(s-\frac{D-1}{2}-\frac{1}{2})\zeta(2s-D)\frac{R^{2s-D+1}}{(1-\sin^2\theta\cos^2\theta)^{s-\frac{D-1}{2}-\frac{1}{2}}}\,\sin\theta$$
$$+\,2^{s-\frac{D-1}{2}+\frac{5}{2}}\,\pi^{s-\frac{D-1}{2}}(\sin\theta)^{s-\frac{D-1}{2}+\frac{1}{2}}\frac{R^{2s-D+1}}{(1-\sin^2\theta\cos^2\theta)^{\frac{1}{2}(s-\frac{D-1}{2}-\frac{1}{2})}}\times$$
$$\sum_{n=1}^{\infty} n^{s-\frac{D-1}{2}-\frac{1}{2}}\cos(2\pi n\cos\theta\sin^2\theta)\sum_{d/n} d^{1-2(s-\frac{D-1}{2})}K_{s-\frac{D-1}{2}-\frac{1}{2}}\Big(\pi n(1-\sin^2\theta\cos^2\theta)^{\frac{1}{2}}\sin\theta\Big)\Bigg)$$

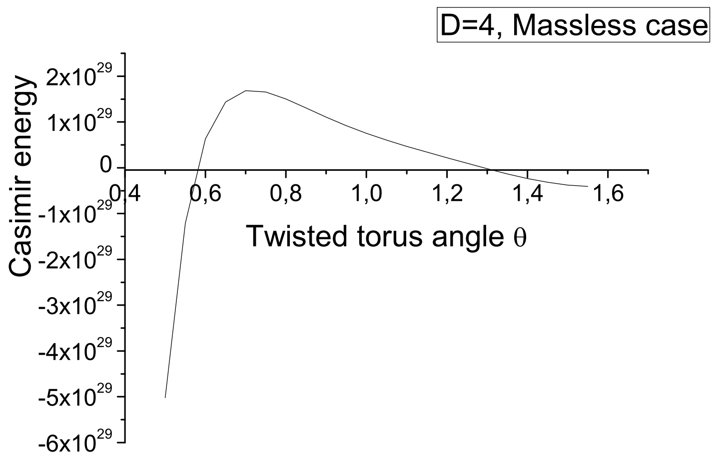


Figure 7: The Casimir energy $\mathcal{E}_c$, for the massless case, as a function of $\theta$, the twisted torus angle, with $D = 4$.

Thus the Casimir force reads,

$$\mathcal{F}_c(s) = -\frac{1}{(2\pi)^{D-1}}\frac{\pi^{\frac{D-1}{2}}}{\Gamma(s)} \times \tag{29}$$
$$\Bigg(2\,\zeta(2s-D+1)2(s-\frac{D-1}{2})R^{2(s-\frac{D-1}{2})-1}\,(\sin^2\theta)^{(s-\frac{D-1}{2})}\,\Gamma(s-\frac{D-1}{2})$$
$$+\,2^{2s-D+1}\sqrt{\pi}\,\Gamma(s-\frac{D-1}{2}-\frac{1}{2})\,\zeta(2s-D)\Big(2s-D+1\Big)\,\frac{R^{2s-D}}{(1-\sin^2\theta\cos^2\theta)^{s-\frac{D-1}{2}-\frac{1}{2}}}\sin\theta$$
$$+\,2^{s-\frac{D-1}{2}+\frac{5}{2}}\,\pi^{s-\frac{D-1}{2}}\Big(2s-D+1\Big)(\sin\theta)^{s-\frac{D-1}{2}+\frac{1}{2}}\,\frac{R^{2s-D}}{(1-\sin^2\theta\cos^2\theta)^{\frac{1}{2}(s-\frac{D-1}{2}-\frac{1}{2})}}\times$$
$$\sum_{n=1}^{\infty} n^{s-\frac{D-1}{2}-\frac{1}{2}}\cos(2\pi n\cos\theta\sin^2\theta)\sum_{d/n} d^{1-2(s-\frac{D-1}{2})}K_{s-\frac{D-1}{2}-\frac{1}{2}}\Big(\pi n(1-\sin^2\theta\cos^2\theta)^{\frac{1}{2}}\sin\theta\Big)\Bigg)$$

In figures 7 and 8 we plot the Casimir energy for $D = 4$ as a function of $\theta$. Also in figures 9 and 10 we present the contour plots of the Casimir energy and Casimir force respectively. The analysis of this case is very similar with the previous analysis. Both the Casimir energy and Casimir force change sign and become positive and negative as $\theta$ varies. As we mentioned previously the most viable case (at least phenomenologically) arises when the Casimir energy is negative and also goes to minus infinity as $R$ decreases (see the contour plot). This is indeed our case and the Casimir energy is negative for some values of $\theta$. For the same values of $\theta$ the force behaves exactly in the same way as the energy does.

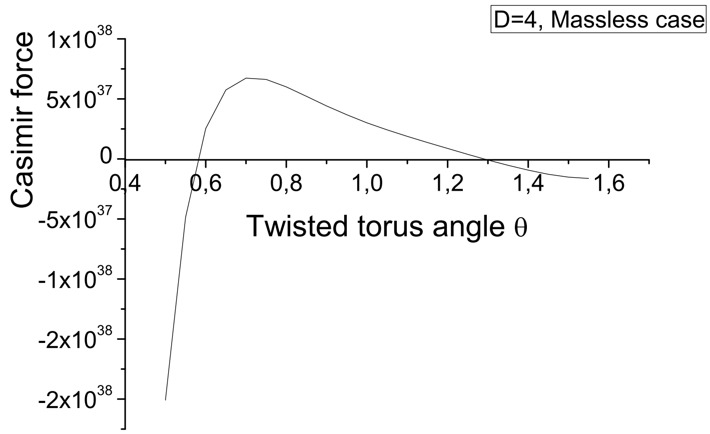


Figure 8: The Casimir force $\mathcal{F}_c$, for the massless case, as a function of $\theta$, the twisted torus angle, for $D = 4$.

We can see that both the force and the energy can take very small positive and negative values and also there exist values of $\theta$ for which both of them are zero.
Another necessary issue to be studied is the stability of the compact dimensions. The shrinking of the compact spaces occurs in this setup but the stability does not. The Casimir energy does not have a stable minimum. We have not studied the $R_1 \neq R_2$ case, which would be more rich in phenomenology but is out of the scope of this article. In the last case the shadowing effects would appear and this would make it an independent exercise, that is to find how the Casimir energy behaves for various dimensions and for the twisted torus for various $\theta$. We shall discuss on these issues in the next section.

# 4 Conclusions

We have studied how the shape of a twisted toroidal compact space can affect the Casimir energy and Casimir force of a scalar field quantized on spacetime of the form $M^D \times T^2_\theta$. Both the massive and massless case were taken into account. The Casimir energy was calculated for general $R_1$, $R_2$ and $\theta$. We specified our study to the case $R_1 = R_2$. The main interest is to see how $\theta$ modifies the Casimir energy and force, so the shape is represented by this parameter. The compact dimensions radius was taken to be $\sim 10^{-8}$, which is compatible with the ADD models predictions and the current experimental bounds.
It was found that for the massive case (we took $mR \ll 1$, but this does not modify the results), the Casimir energy contains infinities for $D$=even. Thus the Casimir energy for massive fields cannot be computed for our spacetime in a consistent way. One could naively say that the singularities could be regularized in some way, however the fact that

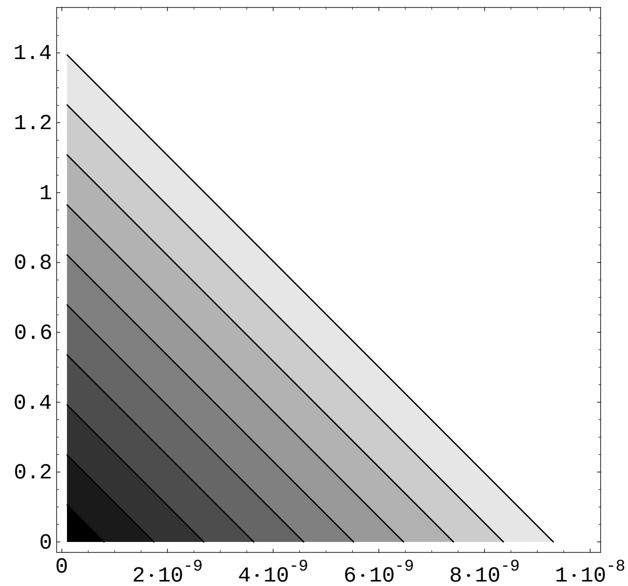


Figure 9: Contour plot of the Casimir energy $\mathcal{E}_c$ as a function of $\theta$ (vertical), the twisted torus angle, and the compact radius $R$ (horizontal), for $D = 4$.

the singularities are $R$-dependent and also the fact that for odd dimensional spaces the Casimir energy is regular, makes us sure that no consistent result for $D$=even holds, at least for the topology of the compact twisted torus. Also the singularities remain when the Casimir force is computed. Thus unfortunately no bulk massive scalar field Casimir energy and force cannot be computed for our spacetime, with this compact extra space. We applied the study for $D = 3$ and $D = 5$ (although this is a mathematical exercise it is interesting to compare it to the massless case where spacetime dimensions $D = 4$ is the only allowed case!). We found similar results both for the two cases. Particularly we found that the energy and force as a function of $\theta$ become positive and negative as $\theta$ varies. Also they can become zero for some $\theta$. Of course the most interesting cases are when the energy and force is negative as we discussed in the previous sections.
In contrast to the massive case, the massless scalar Casimir energy can be computed consistently (that is free of singularities) only for $D = 4$! This is indeed very striking since the $D = 4$ spacetime dimensionality is very peculiar topologically when quantum field theory calculations are performed. We used the same values of $R$ as before and studied how the energy and force behave as a function of $\theta$. We found that the behavior is similar to the massive case, that is, both the energy and force become positive and negative, also for some $\theta$ both take very small values and additionally become zero.
Furthermore we briefly addressed the issue of the stability of the compact space. We saw that the stabilization does not occur for both massive and massless case. This holds of course for the case $R_1 = R_2$. We have not studied the $R_1 \neq R_2$ case because this case

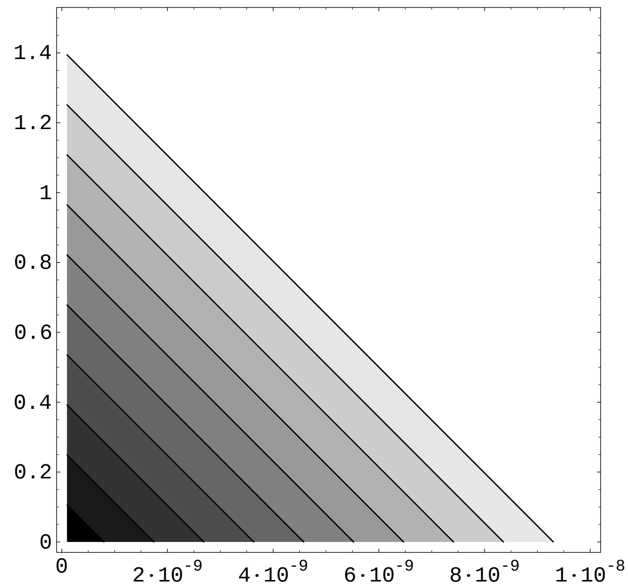


Figure 10: Contour plot of the Casimir force $\mathcal{F}_c$ as a function of $\theta$ (vertical), the twisted torus angle, and the compact radius $R$ (horizontal), for $D = 4$.

deserves a separate study due to the reach phenomenology that exists for this case. Indeed as found by K. Dienes in [2], in this case shadowing effects can take place. Shadowing makes difficult the experimental detection of the exact number and the true geometry of the compact dimensions [2]. Also interesting phenomena arise when $R_1/R_2$ is a rational number. We hope to address this case soon.

Concerning the shadowing effects, when non trivial identifications hold in the compact extra dimensional space, similar to shadowing effect phenomena hold. In reference [17] it was found that the exponential correction to the Newton law force range varies, as the parameters of the identifications change. Thus it is true that we would not be able to be sure on what the number and the geometry of the extra dimensions are.

Before ending we must note the significance of the Casimir energy of bulk fields to cosmology and specifically to dark energy which was addressed in [5] and also in [6]. In [5] the extra compact space had toroidal topology. This corresponds to $\theta = \frac{\pi}{2}$ in our case. It would be interesting to find the cosmological implications of a non-trivial $\theta$ to the dark energy.

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